1.8 Weak containment

Definition 1.8.1. Let Γ be a group, and $\pi : \Gamma \to \mathcal{U}(\mathcal{H})$, $\rho : \Gamma \to \mathcal{U}(\mathcal{K})$ two unitary representations. The representation π is **weakly contained** in the representation ρ (written $\pi \prec \rho$) if for each $\xi \in \mathcal{H}$, $F \subset \Gamma$ finite, and $\varepsilon > 0$, there exists $\eta_1, \ldots, \eta_n \in \mathcal{K}$ such that

$$|\langle \pi(\gamma)\xi, \xi\rangle - \Sigma_{j=1}^{n} \langle \rho(\gamma)\eta_j, \eta_j\rangle| < \varepsilon,$$

for all $\gamma \in F$. The representations π and ρ are weakly equivalent (written $\pi \sim \rho$) if $\pi \prec \rho$ and $\rho \prec \pi$.

It follows easily from the definition that $\pi \sim \pi^{\oplus\infty}$ for any representation π. Also, it is easy to see that containment implies weak containment, and weak containment is a partial order. We also have that if π_i, and ρ_i, $i \in I$ are families of representations such that $\pi_i \prec \rho_i$ for each $i \in I$, then $\oplus_{i\in I}\pi_i \prec \oplus_{i\in I}\rho_i$.

Exercise 1.8.2. Show that if π_1, π_2, ρ_1, and ρ_2 are unitary representations of a group Γ such that $\pi_i \prec \rho_i$, for $i \in \{1, 2\}$, then $\pi_1 \otimes \pi_2 \prec \rho_1 \otimes \rho_2$.

Example 1.8.3. If a representation $\pi : \Gamma \to \mathcal{U}(\mathcal{H})$ weakly contains the trivial representation then for any finite symmetric set $S \subset \Gamma$ and $\varepsilon > 0$ there exists $\eta_1, \ldots, \eta_n \in \mathcal{H}$ such that

$$|1 - \Sigma_{j=1}^{n} \langle \pi(\gamma)\eta_j, \eta_j\rangle| < \varepsilon,$$

for all $\gamma \in S \cup \{e\}$. Hence, if we denote by $\eta = \oplus_{j=1}^{n} \eta_j \in \mathcal{H}^{\oplus n} \subset \mathcal{H}^{\oplus\infty}$, then we have $|1 - \|\eta\|^2| < \varepsilon$, and hence for each $\gamma \in S$ we have

$$\|\eta - \pi^{\oplus\infty}(\gamma)\eta\|^2 = 2(\|\eta\|^2 - \Re(\langle \pi^{\oplus\infty}(\gamma)\eta, \eta\rangle)) < 4\varepsilon.$$

It follows that π contains almost invariant vectors by Lemma 1.5.4.

Conversely, if π contains almost invariant vectors, then it is easy to see that π weakly contains the trivial representation.

The following lemma from [Fel63] is a useful tool for checking if one representation is weakly contained in another.

Lemma 1.8.4. *Let $\pi : \Gamma \to \mathcal{U}(\mathcal{H})$ and $\rho : \Gamma \to \mathcal{U}(\mathcal{K})$ be two unitary representations of a group Γ. Let $\mathcal{L} \subset \mathcal{H}$ be a set such that $\overline{\mathrm{sp}}\pi(\Gamma)\mathcal{L} = \mathcal{H}$. Then $\pi \prec \rho$ if and only if for each $\xi \in \mathcal{L}$, $F \subset \Gamma$ finite, and $\varepsilon > 0$, there exists $\eta_1, \ldots, \eta_n \in \mathcal{K}$ such that*

$$|\langle \pi(\gamma)\xi, \xi\rangle - \Sigma_{j=1}^{n} \langle \rho(\gamma)\eta_j, \eta_j\rangle| < \varepsilon,$$

for all $\gamma \in F$.

Proof. Suppose $\mathcal{L} \subset \mathcal{H}$ is as above, and consider $\mathcal{X} \subset \mathcal{H}$ the set of vectors $\xi \in \mathcal{H}$ such that the positive definite function $\gamma \mapsto \langle \pi(\gamma)\xi, \xi\rangle$ can be approximated arbitrarily well on finite sets by sums of positive definite functions associated to ρ. By hypothesis $\mathcal{L} \subset \mathcal{X}$, and we need to show that $\mathcal{X} = \mathcal{H}$.

If $\xi \in \mathcal{X}$, $\eta \in \mathcal{K}^{\oplus\infty}$, and $\Sigma_{x\in\Gamma}\alpha_x u_x \in \mathbb{C}\Gamma$, then from the formula

$$|\langle \pi(\gamma)(\Sigma_{x\in\Gamma}\alpha_x\pi(x)\xi), \Sigma_{x\in\Gamma}\alpha_x\pi(x)\xi\rangle - \langle \rho(\gamma)(\Sigma_{x\in\Gamma}\alpha_x\pi(x)\eta), \Sigma_{x\in\Gamma}\alpha_x\pi(x)\eta\rangle|$$

$$\leq \Sigma_{x,y\in\Gamma}|\alpha_y\alpha_x||\langle \pi(y^{-1}\gamma x)\xi, \xi\rangle - \langle \rho(y^{-1}\gamma x)\eta, \eta\rangle|,$$

we see that $\Sigma_{x\in\Gamma}\alpha_x\pi(x)\xi \in \mathcal{X}$. In particular, $\mathcal{X}$ is Γ-invariant.

It is also easy to see that $\mathcal{X}$ is a closed set. Moreover, if $\xi, \xi' \in \mathcal{X}$ are such that $\overline{\mathrm{sp}}(\pi(\Gamma)\xi) \perp \overline{\mathrm{sp}}(\pi(\Gamma)\xi')$, then it is easy to see that $\xi + \xi' \in \mathcal{X}$.

In general, we then have that if $\xi, \xi' \in \mathcal{X}$ then

$$\xi + \xi' = (\xi + \mathrm{Proj}_{\overline{\mathrm{sp}}(\pi(\Gamma)\xi)}(\xi')) + (\xi' - \mathrm{Proj}_{\overline{\mathrm{sp}}(\pi(\Gamma)\xi)}(\xi')) \in \mathcal{X}.$$

We therefore have shown that $\mathcal{X}$ is a closed Γ-invariant subspace which contains $\mathcal{L}$ and hence $\mathcal{X} = \mathcal{H}$. □

If $\varphi : \Gamma \to \mathbb{C}$ is a function of positive type and $\pi_\varphi : \Gamma \to \mathcal{U}(\mathcal{H}_\varphi)$ is the corresponding representation described in Section 1.2, then π_φ is generated by a single vector. We therefore obtain the following corollary.

Corollary 1.8.5. *If $\rho : \Gamma \to \mathcal{U}(\mathcal{K})$ is a representation of a group Γ, $\varphi : \Gamma \to \mathbb{C}$ is a function of positive type, and $\pi_\varphi : \Gamma \to \mathcal{U}(\mathcal{H}_\varphi)$ is the corresponding representation. Then $\pi_\varphi \prec \rho$ if and only if $F \subset \Gamma$ finite, and $\varepsilon > 0$, there exists $\eta_1, \dots, \eta_n \in \mathcal{K}$ such that*

$$|\phi(\gamma) - \Sigma_{j=1}^n \langle \rho(\gamma)\eta_j, \eta_j\rangle| < \varepsilon,$$

for all $\gamma \in F$.

Exercise 1.8.6. Suppose $\Gamma \curvearrowright X$ is an action of a group Γ on a set X, and $\alpha : \Gamma \times X \to \Lambda$ is a cocycle into a group Λ. If $\pi : \Lambda \to \mathcal{U}(\mathcal{H})$, and $\rho : \Lambda \to \mathcal{U}(\mathcal{K})$ are two representations such that $\pi \prec \rho$, then show that $\mathrm{Ind}_\Lambda^\alpha \pi \prec \mathrm{Ind}_\Lambda^\alpha \rho$.

Conclude that if $\Sigma < \Gamma$ is an amenable subgroup of a group Γ then $\lambda_{\Gamma/\Sigma} \prec \lambda_\Gamma$.

For further properties of weak containment a good place to look is Appendix F in [BdlHV08].

Definition 1.8.7. [Bek90] Let $\pi : \Gamma \to \mathcal{U}(\mathcal{H})$ be a unitary representation of a group Γ, then π is **amenable** if there exists a tate $\Phi \in (\mathcal{B}(\mathcal{H}))^*$ such that $\Phi(\pi(\gamma)T) = \Phi(T\pi(\gamma))$ for all $\gamma \in \Gamma$, $T \in \mathcal{B}(\mathcal{H})$.

Note that Γ is amenable if and only if the left-regular representation is amenable. We also have an analogue of Theorem 1.6.5 for amenable representations. The proof is similar, however we will not present it here.

Theorem 1.8.8. *[Bek90] Let $\pi : \Gamma \to \mathcal{U}(\mathcal{H})$ be a unitary representation of a group Γ, then the following conditions are equivalent.*

(1). *π is amenable.*

(2). *There exists a net of trace class operators* $T_i \in \mathcal{B}(\mathcal{H})$ *such that* $\|T_i\|_{\mathrm{Tr}} = 1$, *and* $\|T_i\pi(\gamma) - \pi(\gamma)T_i\|_{\mathrm{Tr}} \to 0$, *for all* $\gamma \in \Gamma$.

(3). *There exists a net of finite rank projections* $P_i \in \mathcal{B}(\mathcal{H})$ *such that* $\frac{1}{\|P_i\|_{\mathrm{HS}}}\|P_i\pi(\gamma) - \pi(\gamma)P_i\|_{\mathrm{HS}} \to 0$, *for all* $\gamma \in \Gamma$.

(4). $\pi \otimes \overline{\pi}$ *contains almost invariant vectors.*

(5). *For any finite symmetric set* $S \subset \Gamma$ *the operator* $T_S = \frac{1}{|S|}\Sigma_{\gamma\in S}\pi \otimes \overline{\pi}(\gamma)$ *satisfies* $\|T_S\| = 1$.

Bibliography

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